B.Sc. Part 2 (Hons.) 4^{th} Paper

Diff. Equations (orthogonal trajectories contd.)

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Some important Sums

V.V. Impt I. Prove that the system of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self-orthogonal.

Soln Given that

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{—— (1)}$$

Differentiating with respect to x, we get

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}p = 0 \quad \left[\because p = \frac{dy}{dx}\right]$$

$$\Rightarrow \frac{x}{a^2+\lambda} = \frac{-py}{b^2+\lambda} \Rightarrow \frac{a^2+\lambda}{x} = \frac{b^2+\lambda}{-py}$$

$$\Rightarrow x(b^2+\lambda) + py(a^2+\lambda) = 0$$

$$\Rightarrow \lambda(x+py) + (xb^2+pya^2) = 0$$

$$\Rightarrow \lambda = -\left(\frac{xb^2+pya^2}{x+py}\right)$$

$$\therefore a^2+\lambda = a^2 - \frac{b^2x+pya^2}{x+py}$$

$$\Rightarrow a^2+\lambda = \frac{a^2x-b^2x}{x+py} = \frac{x(a^2-b^2)}{x+py} \quad \text{--- (2)}$$

and $$b^2+\lambda = b^2 - \frac{xb^2+pya^2}{x+py}$$

$$\Rightarrow b^2+\lambda = \frac{pb^2y-pya^2}{x+py} = \frac{py(b^2-a^2)}{x+py} \quad \text{--- (3)}$$

Putting the values from (2) and (3) in (1),

we get

$$\frac{x^2(x+py)}{x(a^2-b^2)} + \frac{y^2(x+py)}{py(b^2-a^2)} = 1$$

$$\Rightarrow \frac{x(x+py)}{a^2-b^2} - \frac{y(x+py)}{p(a^2-b^2)} = 1$$

$$\Rightarrow px(x+py) - y(x+py) = p(a^2-b^2)$$

$$\Rightarrow (px-y)(x+py) = p(a^2-b^2) \quad \text{--- (4)}$$

Replacing $\frac{dy}{dx}$ by $\frac{-1}{dy/dx}$ i.e. p by $-\frac{1}{p}$ in above,

$$\left(\frac{-x}{p}-y\right)\left(x+\frac{-y}{p}\right) = \frac{-1}{p}(a^2-b^2)$$

$$\Rightarrow -(x+py)(px-y) = -p(a^2-b^2) \Rightarrow (x+py)(px-y) = p(a^2-b^2)$$

which is same as (4).

Hence the given system is self-orthogonal.

Q. Find the orthogonal trajectories of the family of curves

$$x^2+y^2+2gx+c=0$$

Soln.

~~Di~~ Given eqn.

$$x^2+y^2+2gx+c=0 \quad \text{--- (1)}$$

Diff. w.r.to x, we get

$$2x+2y\frac{dy}{dx}+2g=0$$

$$\Rightarrow x+py+g=0 \Rightarrow g=-(x+py)$$

So (1) becomes

$$x^2+y^2-2x(x+py)+c=0$$

$$\Rightarrow y^2-x^2-2pxy+c=0 \quad \text{--- (2)}$$

For orthogonal trajectories, ~~put~~ replace p by $-\frac{1}{p}$ in (2), we get

$$y^2-x^2+\frac{2xy}{p}+c=0$$

$$\cancel{\Rightarrow p(y^2-x^2)+2xy+cp=0}$$

$$\cancel{\Rightarrow \frac{dy}{dx}}$$

$$\Rightarrow y^2-x^2+2xy\frac{dx}{dy}+c=0$$

$$\Rightarrow 2xy\frac{dx}{dy}-x^2=-c-y^2 \quad \text{--- (3)}$$

Put $x^2 = z \Rightarrow 2x\frac{dx}{dy} = \frac{dz}{dy}$

So, (3) becomes

$$y \cdot \frac{dz}{dy} - z = -c - y^2$$

$$\Rightarrow \frac{dz}{dy} - \frac{z}{y} = \frac{-c}{y} - y.$$

which is a linear eqn.

$$I.F. = e^{-\int \frac{1}{y}dy} = e^{-\log y} = e^{\log y^{-1}} = \frac{1}{y}.$$

∴ Soln is given by

$$z \times IF = \int Q \times IF \, dy$$

$$\Rightarrow z \times \frac{1}{y} = \int \frac{1}{y}\left(-\frac{c}{y} - y\right)dy$$

$$\Rightarrow \frac{x^2}{y} = \int \left(-\frac{c}{y^2} - 1\right) dy$$

$$\Rightarrow \frac{x^2}{y} = +\frac{c}{y} - y + K$$

$$\Rightarrow x^2 = c - y^2 + Ky$$

$$\Rightarrow x^2 + y^2 - Ky - c = 0$$

This is the reqd. orthogonal trajectory.